The Cross Ratio

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Let $z_2, z_3, z_4 \in \hat{\mathbb{C}}$ - three different points.

<u>Theorem</u> $\exists!$ Möbius S: $Sz_2 = 1$, $Sz_3 = 0$, $Sz_4 = \infty$.

<u>Proof.</u> If $z_2 \neq \infty$, $z_3 \neq \infty$, $z_4 \neq \infty$:

$$S(z) = \frac{z - z_3}{z - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}$$

$z_2 = \infty$: $\quad S(z) = \frac{z - z_3}{z - z_4}$

$z_3 = \infty$: $\quad S(z) = \frac{z_2 - z_4}{z - z_4}$

$z_4 = \infty$: $\quad S(z) = \frac{z - z_3}{z_2 - z_3}$

<u>Uniqueness:</u> S_1 - another such transformation.

Then $SS_1^{-1}(0) = 0$, $SS_1^{-1}(1) = 1$, $SS_1^{-1}(\infty) = \infty$

$\Downarrow$ $\qquad \Downarrow \qquad \Downarrow$

$b = 0 \Longrightarrow a = d \Longleftarrow c = 0$

$\Downarrow$

$SS_1^{-1}(z) = z$. ∎

<u>Def.</u> The cross ratio $(z_1, z_2, z_3, z_4) = S(z_1)$, where S - Möbius map with $Sz_2 = 1$, $Sz_3 = 0$, $Sz_4 = \infty$.

<u>Theorem.</u> Let T be a Möbius map. Then

(Cross ratio is invariant)

$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$ for any four distinct $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$.

<u>Proof.</u> Let $Sz_2 = 1$, $Sz_3 = 0$, $Sz_4 = \infty$.

Then $ST^{-1}(Tz_2) = 1$, $ST^{-1}(Tz_3) = 0$, $ST^{-1}(Tz_4) = \infty$

So $(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}(Tz_1) = Sz_1 = (z_1, z_2, z_3, z_4)$ ∎

<u>Theorem</u> The cross ratio $(z_1, z_2, z_3, z_4) \in \mathbb{R} \iff$ z_1, z_2, z_3, z_4 lie on the same circle or line.

<u>Proof.</u> $(z_1, z_2, z_3, z_4) \in \mathbb{R} \iff (Sz_1, Sz_2, Sz_3, Sz_4) \in \mathbb{R} \iff (Sz_1, 1, 0, \infty) = Sz_1 \in \mathbb{R}$

Take the right S

$\iff Sz_1$ lies on the line generated by $0 = Sz_2$, $1 = Sz_3$, $\infty = Sz_4$

lines and circles are Möbius invariant $\iff$ z_1 lies on the same line or circle as z_2, z_3, z_4 ∎

<u>Symmetry.</u>

Symmetry with respect to $\mathbb{R}$: $z \to \bar{z}$

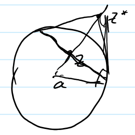
$$(z, 1, 0, \infty) \longrightarrow (\bar{z}, 1, 0, \infty) = \overline{(z, 1, 0, \infty)}$$

Def. The points z and z^* are symmetric with respect to the line or the circle generated by z_2, z_3, z_4 if $(z, z_2, z_3, z_4) = \overline{(z^*, z_2, z_3, z_4)}$.
(or $Sz = \overline{Sz^*}$).

Theorem. Does not depend on the choice of z_2, z_3, z_4 on the same line or circle.

Symmetric wrt a line: the usual symmetry.

Symmetric wrt a circle $|z-a|^2 = r^2$: $(z^*-a)(\bar{z}-\bar{a}) = r^2$.



$$\frac{|z^*-a|}{r} = \frac{r}{|z-a|}$$
$$\arg(z^*-a) = \arg(z-a).$$

Proof. First, let us observe that

if $z_2, z_3, z_4 \in \mathbb{R}$, then $z^* = \bar{z}$ (the map S has real coefficients, so $\frac{a\bar{z}+b}{c\bar{z}+d} = (\bar{z}, z_2, z_3, z_4) = \overline{\left(\frac{az+b}{cz+d}\right)} = \overline{(z, z_2, z_3, z_4)}$)

if z_2, z_3, z_4 lie on the same line, then $\exists\, Tz = az+b$, such that $Tz_2, Tz_3, Tz_4 \in \mathbb{R}$. T preserves symmetry and crossratio.

Finaly, if $z_2, z_3, z_4 \in \{|z-a| = r^2\}$ i.e. $(z_j - a)(\bar{z}_j - \bar{a}) = r^2$, then
$\overline{(z, z_2, z_3, z_4)} = \overline{(z-a, z_2-a, z_3-a, z_4-a)} = (\bar{z}-\bar{a}, \bar{z}_2-\bar{a}, \bar{z}_3-\bar{a}, \bar{z}_4-\bar{a}) =$
$(\bar{z}-\bar{a}, \frac{r^2}{z_2-a}, \frac{r^2}{z_3-a}, \frac{r^2}{z_4-a}) = (\frac{r^2}{\bar{z}-\bar{a}}, z_2-a, z_3-a, z_4-a) =$
$(\underbrace{\frac{r^2}{\bar{z}-\bar{a}} + a}_{z^*}, z_2, z_3, z_4)$

$(z^*-a)(\overline{z-a}) = r^2 = (\overline{z^*-a})(z-a)$ ∎

Remark. With respect to $\{|z-a| = r\}$, $a^* = \infty$

Proof Take $z_2 = a+r$, $z_3 = a-r$, $z_4 = a+ir$

Then $(\infty, z_2, z_3, z_4) = \frac{z_2-z_4}{z_2-z_3} = \frac{r-ir}{2r} = \frac{1-i}{2}$.

$(a, z_2, z_3, z_4) = \frac{a-z_3}{a-z_4} \frac{z_2-z_4}{z_2-z_3} = \frac{r}{-ir} \cdot \frac{1-i}{2} = i\,\frac{1-i}{2} = \frac{1+i}{2} = \overline{(\infty, z_2, z_3, z_4)}$ ∎

Corollary T – Möbius, z, z^* – symmetric with respect a circle or line $\ell \Rightarrow Tz, Tz^*$ – symmetric wrt circle or line $T\ell$.

Proof. $z_2, z_3, z_4 \in \ell$ $\quad (z, z_2, z_3, z_4) = \overline{(z^*, z_2, z_3, z_4)}$
$$\Updownarrow$$
$(Tz, Tz_2, Tz_3, Tz_4) = \overline{(Tz^*, Tz_2, Tz_3, Tz_4)}$ ∎

Theorem. T – Möbius, $T(\mathbb{D}) = \mathbb{D}$ $\quad (\mathbb{D} = B(0,1))$.

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Then $Tz = e^{i\theta}\dfrac{z-a}{1-\bar{a}z}$, for some $a \in \mathbb{D}$, $\theta \in \mathbb{R}$.

Proof. First, for $|z| = 1$, $z = e^{i\varphi}$

$$|Tz| = |e^{i\theta}| \frac{|z-a|}{|1-\bar{a}z|} = \frac{|z-a|}{|z|\,|\bar{z}-\bar{a}|} = 1.$$ So the circle is preserved.

So $\mathbb{D}$ is mapped either to itself, or to $\mathbb{D}_- = \{|z|>1\}$.

But $a \to 0$, $a \in \mathbb{D}$, so $T\mathbb{D} = \mathbb{D}$.

Let $T(\mathbb{D}) = \mathbb{D}$. Then let $a = T^{-1}0 \in \mathbb{D}$.

Then $T(a^*) = T(1/\bar{a}) = 0^* = \infty$.

so
$$T(z) = c\,\frac{z-a}{z-\frac{1}{\bar{a}}} = \underbrace{-c\bar{a}}_{d}\,\frac{z-a}{1-\bar{a}z} = d\,\frac{z-a}{1-\bar{a}z}.$$

But $|T1| = 1$, so $|d|\dfrac{|1-a|}{|1-\bar{a}1|} = 1 \Rightarrow |d| = 1 \Rightarrow d = e^{i\theta}$ $\blacksquare$

Theorem $\mathbb{H} = \{\operatorname{Im} z > 0\}$. $T\mathbb{H} = \mathbb{H} \Longleftrightarrow T = \dfrac{az+b}{cz+d}$ $\quad a, b, c, d \in \mathbb{R}$.

$ad - bc > 0$.

Proof. Very similar, left as exercise $\blacksquare$

$$\operatorname{Im} \frac{ai+b}{ci+d} > 0$$